Why Quadratic Mean Diameter?

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ABSTRACT: *Quadratic mean diameter is the measure of average tree diameter conventionally used in forestry, rather than arithmetic mean diameter. The historical and practical reasons for this convention are reviewed. West. J. Appl. For. 15(3):137–139.*

Average diameter is a widely used stand statistic that appears in virtually all yield tables, simulator outputs, stand summaries, and much inventory data. To most people, "average" is synonymous with the arithmetic mean, defined as

$$\text{arithmetic mean} = \bar{x} = (\Sigma\ x_i) / n$$

where the x_i are the individual measurements and n is the total number of measurements.

But there are in fact some half dozen different kinds of averages (= means = measures of central tendency), each appropriate to specific uses. One of these is the quadratic mean (Kendall and Buckland 1967, Iles and Wilson 1977), defined as

$$\text{quadratic mean} = \sqrt{(\Sigma x_i^2) / n}$$

which is the square root of the arithmetic mean of squared values. Other generally recognized means sometimes encountered in forestry are the geometric mean, harmonic mean, median, and mode.

The expression of average stand diameter conventionally used in forestry is not the arithmetic mean of diameters, but the quadratic mean,

$$\text{quadratic mean diameter} = \sqrt{(\Sigma d_i^2) / n}$$

where d_i is the diameter at breast height of an individual tree, and n is the total number of trees. Quadratic mean diameter is commonly symbolized as *QMD*, *Dq,* or *Dg*. *Dg*, in which the subscript stands for "Grundfläche," German for basal area, is widely used in Europe and is the symbol recommended by the International Union of Forest Research Organizations (Van Soest et al. 1959).

QMD is often calculated by the equivalent equation:

$$QMD = \sqrt{B / (k * n)}$$

where B is stand basal area, n is corresponding number of trees, and k is a constant that depends on the measurement units used (0.005454 for B in square feet and *QMD* in inches; 0.0000785 for B in square meters and *QMD* in centimeters).

In angle-gauge sampling, it can also be calculated directly (Buckingham 1969) as:

$$QMD = \sqrt{[n_s / \Sigma(1 / d_i^2)]}$$

where the d_i are the diameters and n_s is the number of "in" trees in the angle-gauge sample.

Past usage of the phrase "average diameter" has often been very loose, and unwary readers often take it to mean the arithmetic mean, when in fact the value given is the quadratic mean. It is therefore good practice for authors to be specific (Curtis 1968). The quadratic mean gives greater weight to larger trees and is equal to or greater than the arithmetic mean by an amount that depends on the variance according to the relationship

$$(QMD)^2 = \bar{d}^2 + s^2$$

where $\bar{d}$ is arithmetic mean diameter and s^2 is the variance of diameters.

In stands of small diameter and narrow range in diameters, the differences are slight. In stands with large diameters and a wide range of diameters present or with strongly skewed diameter distributions, the differences between arithmetic mean and quadratic mean diameters can be substantial (Figure 1).

People not strongly grounded in forest mensuration are often unaware of the distinction between arithmetic and quadratic means. When this distinction is pointed out, they naturally wonder how and why such a strange "average" came to be adopted and why it is still used. After all, it is rarely mentioned in introductory statistics courses.

The answer is partly a matter of custom and historical precedent, but *QMD* also has certain practical advantages that still hold true.

Use of the quadratic mean of diameters is a very old practice in forestry, which goes back to 19th century Germany and possibly earlier. It has been standard practice in the United States from the earliest days of North American forestry. Most standard U.S. mensuration texts,

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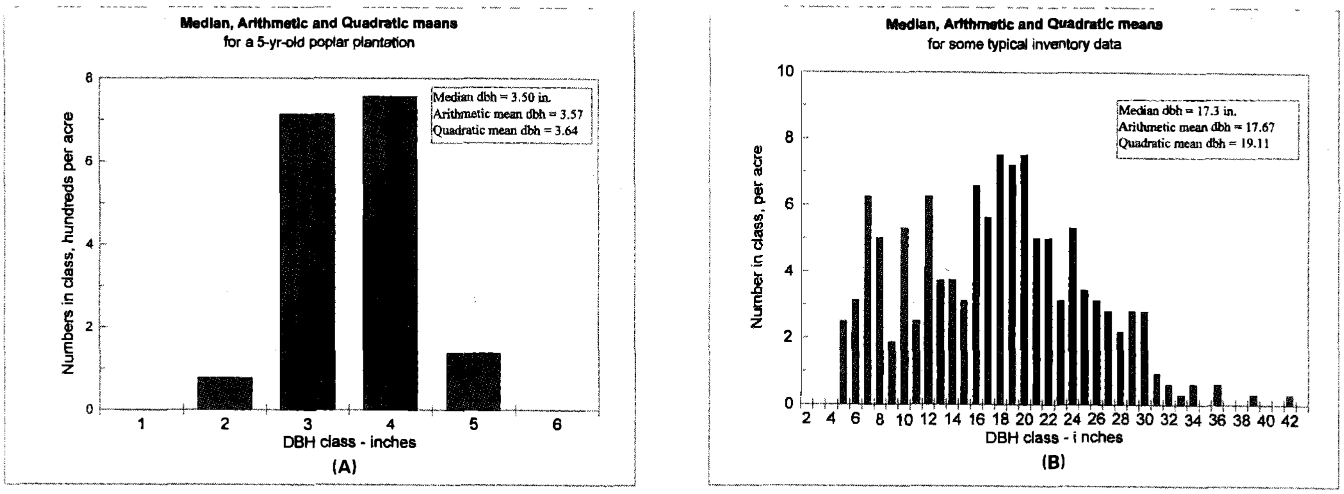

Figure 1. Median, arithmetic mean, and quadratic mean diameters for stands with (A) small diameters and nearly symmetrical diameter distribution, and (B) larger diameters and somewhat asymmetrical diameter distribution.

starting with Graves (1908), define average stand diameter as the diameter corresponding to the tree of arithmetic mean basal area, which *is* the quadratic mean diameter. Braathe's (1957) summary of European thinning literature specifically defines average diameter as the quadratic mean. *QMD* is commonly used in silviculture research data summaries and reports. Virtually all normal yield tables prepared in the United States in the period from around 1920 through the mid-1960s use quadratic mean diameters (Schnur 1937, McArdle et al. 1961, Barnes 1962), sometimes referred to in older publications as "average diameter by basal area." This usage of *QMD* is also common in current stand simulation programs (Curtis et al. 1981, Hann et al. 1997). Reineke's (1933) SDI is based on *QMD*, as are the various relative density measures and stand management diagrams derived from the Reineke relationship (Curtis 1982, Long et al. 1988).

In the Germany of some 150 or more years ago, there were a number of so-called mean tree methods in use for estimating volume of wood in forest stands. These also had some limited use in the early days of North American forestry (Graves 1908, p. 224ff.). The basic idea, in simplest form, was that the forester would select a tree(s) considered average for the stand, cut it and measure its wood content, and then multiply by the number of trees. The obvious difficulty was in selecting an average tree(s), whose volume would approximate overall arithmetic mean volume / tree. In regular even-aged stands, diameter of the tree of arithmetic mean volume is generally close to that of the tree of arithmetic mean basal area (which is also the tree of quadratic mean diameter). Thus, a basis was provided for selecting sample trees for analysis.

Such procedures are now ancient history. But justification for use of *QMD* also arises from the general relationship between stand volume and other, directly measurable, stand attributes.

In any reasonably regular stand, there is a general relationship

$$\text{volume / unit area} = f * N * B_{mn} * H$$

where f = stand form factor, which for a given species and stand condition has only a very limited range of variation and can often be treated as constant.

N = number of trees / unit area

B_{mn} = arithmetic mean basal area / tree, and

H = some "average" height.

In an existing stand we cannot directly measure either total stand volume or mean volume/tree, but must estimate these from measurements of their components. It is often convenient to describe stands in terms of means of these components: namely, number of trees, arithmetic mean basal area, and some average height.

People do not usually think in terms of basal area of a tree (cross-sectional area at breast height). It is much easier to visualize a tree of 19 in. dbh than one of 2.0 ft^2 cross-sectional area. It is therefore common to describe stands by *QMD* (a surrogate for arithmetic mean basal area) rather than by arithmetic mean basal area. In these terms,

$$\text{Volume / unit area} = f * N * [k * (QMD)^2] * H$$

The correct average height in these equations is not the arithmetic mean, but Lorey's height (H_L, named after a 19th century German forester). This is a weighted mean,

$$H_L = \Sigma b_i h_i \,/\, \Sigma b_i = \Sigma d_i^2 h_i \,/\, \Sigma d_i^2$$

where, b_i is the basal area of an individual tree and d_i is the diameter of an individual tree.

H_L is somewhat inconvenient to calculate from a fixed area sample or stand table, although with angle-gauge sampling it can be easily obtained as the arithmetic mean of heights of the count trees. A common approximation is the height corresponding to *QMD*, as estimated by a height-diameter curve or equation for the individual stand. (Stand average height is of course a different statistic from the top height or dominant height used for other purposes, though highly correlated with these in unthinned stands.)

Expressions of the above form are not commonly used today to calculate stand volumes, although they are valid and are sometimes used in stand simulation programs. We generally apply tree volume equations directly and sum over all trees, rather than first calculating these means. But there is another strong reason for using *QMD*. This stems from the relationship

$$B = k * N * (QMD)^2$$

This is an exact relationship. Therefore, knowledge of any two of the variables automatically confers knowledge of the third. In contrast, there is no equivalent exact relationship for the arithmetic mean, and conversions using the arithmetic mean also require knowledge of the variance. It is a great deal easier to make consistent estimates and projections for two variables than for three, and the exact relationship that exists when *QMD* is used markedly simplifies construction of yield tables and stand simulators, stand projections, and some inventory computations.

This direct convertibility also simplifies the construction and use of stand management diagrams based on number of trees, basal area, and average diameter (Long et al. 1988, Gingrich 1967, Ernst and Knapp 1985). Because of this convertibility, they can be expressed in terms of any two of the three variables *N*, basal area, and *QMD*.

The arithmetic mean is the measure of central tendency most widely used in general statistics, and is essential to a few procedures (such as defining a normal probability distribution). But most procedures in common use in forestry do not specifically require the use of the arithmetic mean. Both the mensurational advantages mentioned above and long-standing precedent make the quadratic mean of diameters the preferred "average diameter" for expressing stand attributes. In any case, users should be conscious of the difference between quadratic and arithmetic mean diameters (which usually is not large) and be specific in defining the value used.

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